Ranking voting systems and surrogate weights: Explicit formulas for centroid weights

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Abstract

One of the most important issues in the field of ranking voting systems is the choice of the weighting vector. This issue has been addressed in the literature from different approaches, and one of them has been to obtain the weighting vector as a solution to a linear programming problem. In this paper we analyze some models proposed in the literature and show that one of their main shortcomings is that they cannot guarantee the uniqueness of the solution, so the winner or the final ranking of the candidates may depend on the chosen weighting vector. An alternative to these models is the use of surrogate weights, among which rank order centroid (ROC) weights stand out as the centroid of a specific simplex. Following this idea, in this paper we show the explicit expression for the weights that form the centroid of diverse simplices utilized in ranking voting systems, and we also see that certain surrogate weights frequently employed in literature can be derived as extreme cases where the simplices collapse into a single vector. Moreover, we argue that averaging two weighting vectors can be a valid approach in some cases and, in this way, we can get weighting vectors that closely resemble those used in some sports competitions.

Keywords: Decision support systems, ranking voting systems, weighting vectors, surrogate weights, centroid.

1. Introduction

A classical problem in the field of decision making is to obtain a winning candidate or a global ranking of a set of candidates (or alternatives) from the individual rankings given by a set of voters. This problem has been widely addressed in the field of social choice, especially since Arrow's famous theorem (Arrow (1963)), and among the great diversity of proposed methods, two trends can be highlighted (see, for instance,

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Brams & Fishburn (2002)). On the one hand, there are methods that satisfy the Condorcet winner criterion, that is, those methods that choose the Condorcet winner when one exists. On the other hand, there are positional methods, that is, those in which the winners are selected using the positions of the candidates in the voters' preference orders (see Llamazares & Peña (2015a) and the references therein). Among positional methods, scoring rules deserve special attention. In these functions, fixed scores are assigned to the different ranks obtained by the candidates and these ones are ordered according to the total number of points they receive. To be more specifically, given $\{A_1, \ldots, A_n\}$ a set of n candidates, suppose that each voter selects m candidates and ranks them from top to mth place. The score obtained by the candidate A_i is $Z_i = \sum_{j=1}^{m} v_{ij} w_j$, where v_{ij} is the number of jth place ranks that candidate A_i occupies and w_j is the score given to the jth place.

Obviously the choice of the scoring vector $(w_1, \ldots, w_m)$ may determine the winning candidate (or the global ranking). To avoid the arbitrariness inherently involved in choosing the scoring vector, some methods have been proposed based on considering certain uncertainty about the weights (see, for instance, Viappiani (2020)). Some of the pioneers in this field were Cook & Kress (1990), who proposed using Data Envelopment Analysis (DEA) to determine the most favourable scoring vector for each candidate. The model DEA/AR proposed by these authors is the following:

$$
\begin{aligned}
&\text{Maximize } Z_o = \sum_{j=1}^{m} v_{oj} w_j, \\
&\qquad \text{s.t. } \sum_{j=1}^{m} v_{ij} w_j \leq 1, \quad i = 1, \ldots, n, \\
&\qquad\qquad w_j - w_{j+1} \geq d(j, \varepsilon), \quad j = 1, \ldots, m-1, \\
&\qquad\qquad w_m \geq d(m, \varepsilon),
\end{aligned}
\tag{1}
$$

where $\varepsilon \geq 0$ and the functions $d(j, \varepsilon)$, called the *discrimination intensity functions*, are nonnegative and nondecreasing in ε. Furthermore, $d(j, 0) = 0$ for all $j \in \{1, \ldots, m\}$.

However, one important shortcoming of their model is that several candidates are often efficient, i.e., they achieve the maximum attainable score. For this reason, various alternative models have emerged in the literature to address this kind of problems. Most of the proposed models have followed Cook and Kress's original idea of allowing each candidate to be evaluated with the scoring vector that favors them the most (see, for instance, Green et al. (1996), Hashimoto (1997), Obata & Ishii (2003), Foroughi & Tamiz (2005), and Kim & Ahn (2022)), and sometimes consider both an optimistic and pessimistic viewpoint (see Khodabakhshi & Aryavash (2015), Llamazares (2017), and Ahn et al. (2019)). However, most of the proposed models present a very significant shortcoming: the relative order between two candidates may change when the number of first, second, ..., mth ranks obtained by other candidates changes, although there is no variation in the number of first, second, ..., mth ranks obtained by both candidates (see Llamazares & Peña (2009)), which makes it difficult for such models to be put into practice.

Alongside the aforementioned approach, other models emerged where the goal was to find a single scoring vector to evaluate all the candidates. Two linear programming models that have been widely studied in the literature are the ones proposed by Wang et al. (2007). For instance, Ahn (2017a) gave closed expressions for the optimal solutions of both models by using the extreme points of the simplex obtained from the constraints on the weights. However, as we will see later, these optimal solutions are not valid in all cases, although in the first of the models it is possible to establish a condition that allows guaranteeing the validity of the result given by Ahn (2017a). In another study, the analysis conducted by Foroughi & Aouni (2012) has revealed that both models present some significant drawbacks, leading these authors to suggest their own models. However, the main issue of these latter models is that the uniqueness of the solutions cannot be guaranteed, meaning that the winner or the final ranking of the candidates may depend on the chosen weighting vector.

An alternative to these models is employing surrogate weights, with ROC weights being particularly notable as the centroid of a specific simplex. Building on this idea, our paper presents the explicit expression for the weights that form the centroid of diverse simplices utilized in ranking voting systems. As particular cases of the obtained formulas, we show that some of the most commonly used surrogate weights can be seen as extreme cases where the simplices collapse into a single vector. Besides, we also justify that in some cases it makes sense to consider scoring vectors obtained as averages of two others, and that this procedure allows us to obtain weighting vectors that bear striking resemblance to those employed in certain sports contests.

The rest of the paper is organised as follows. In Section 2, we review some notable results regarding the extreme points of certain simplices. Section 3 is dedicated to two linear models proposed by Wang et al. (2007) and the analysis of the solutions provided by Ahn (2017a). In Section 4 we analyze the models suggested by Foroughi & Aouni (2012), which were introduced to address the issues presented by the models of Wang et al. (2007). Section 5 is devoted to surrogate weights and, in addition to obtaining a general expression for the centroid of certain simplices, we present some specific cases of special interest. Finally, some concluding remarks are made in Section 6.

2. Preliminaries

In this section we recall some interesting results given by Carrizosa et al. (1995) and Mármol et al. (1998) on the extreme points of the simplices

$$S_{(\boldsymbol{A},\boldsymbol{\lambda})} = \{\boldsymbol{w} \in \mathbb{R}^m \mid \boldsymbol{w} \geq \boldsymbol{0}, \boldsymbol{A}\boldsymbol{w} \geq \boldsymbol{\lambda},\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{w} = 1\},$$

where $\boldsymbol{A}$ is an invertible matrix of order m such that $\boldsymbol{A}^{-1} = \boldsymbol{H} = (h_{ij})$ is componentwise nonnegative (that is, $h_{ij} \geq 0$ for all $i, j \in \{1, \ldots, m\}$), $\boldsymbol{\lambda} \geq \boldsymbol{0}$, and $\boldsymbol{e}^{\mathrm{T}} = (1, \ldots, 1)$.

Remark 1. Notice that $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ is not empty if and only if $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{\lambda} \leq 1$; and that if $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{\lambda} = 1$, then $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ has only one vector, $\boldsymbol{A}^{-1}\boldsymbol{\lambda}$ (see Corollary 3 in Mármol et al. (1998)).

The following theorem shows the extreme points of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ as the columns of a matrix constructed from $\boldsymbol{A}^{-1}$ and λ (see Carrizosa et al. (1995); Mármol et al. (1998)).

Theorem 1. *Let $\boldsymbol{A}$ be an invertible matrix of order m and $\boldsymbol{\lambda} \geq \boldsymbol{0}$. If $\boldsymbol{A}^{-1}$ is componentwise nonnegative and $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{\lambda} < 1$, then the extreme points of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ are the columns of the matrix $\boldsymbol{A}^{-1}\boldsymbol{D}$, where*[1]

$$\boldsymbol{D} = \begin{pmatrix} \mu_1^{-1}(1 - \sum_{k\neq 1}\mu_k\lambda_k) & \lambda_1 & \ldots & \lambda_1 \\ \lambda_2 & \mu_2^{-1}(1 - \sum_{k\neq 2}\mu_k\lambda_k) & \ldots & \lambda_2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_m & \lambda_m & \ldots & \mu_m^{-1}(1 - \sum_{k\neq m}\mu_k\lambda_k) \end{pmatrix}$$

and μ_k ($k = 1, \ldots, m$) is the sum of the elements of the kth column of $\boldsymbol{A}^{-1}$.

Remark 2. Notice that the matrix $\boldsymbol{D}$ can be also written as

$$\boldsymbol{D} = \begin{pmatrix} \lambda_1 + \mu_1^{-1}\delta & \lambda_1 & \ldots & \lambda_1 \\ \lambda_2 & \lambda_2 + \mu_2^{-1}\delta & \ldots & \lambda_2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_m & \lambda_m & \ldots & \lambda_m + \mu_m^{-1}\delta \end{pmatrix}, \tag{2}$$

where μ_k ($k = 1, \ldots, m$) is the sum of the elements of the kth column of $\boldsymbol{A}^{-1}$ and $\delta = 1 - \sum_{k=1}^{m}\mu_k\lambda_k$. Besides, since $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}^{-1} = (\mu_1, \ldots, \mu_m)$, $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{\lambda} = \sum_{k=1}^{m}\mu_k\lambda_k$, and $\delta = 1 - \boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{\lambda}$, we have that $\delta = 0$ if and only if $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{\lambda} = 1$ and, in this case, all the columns of the matrix $\boldsymbol{A}^{-1}\boldsymbol{D}$ are equal to $\boldsymbol{A}^{-1}\boldsymbol{\lambda}$.

[1] There is a misprint in the expression of the matrix $\boldsymbol{D}$ given by Carrizosa et al. (1995) (see Corollary 4.1 of that paper).

In the specific case of $\boldsymbol{\lambda} = \mathbf{0}$ we get the following result (Carrizosa et al. (1995)).

Corollary 1. *Let $\boldsymbol{A}$ be an invertible matrix of order m. If $\boldsymbol{A}^{-1}$ is componentwise nonnegative then the extreme points of $S_{(\boldsymbol{A},\mathbf{0})} = \{\boldsymbol{w} \in \mathbb{R}^m \mid \boldsymbol{w} \geq \mathbf{0}, \boldsymbol{A}\boldsymbol{w} \geq \mathbf{0},\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{w} = 1\}$ are the columns of $\boldsymbol{A}^{-1}$ normalized to add 1.*

In Sections 3 and 4 of this paper we analyze some models proposed by Wang et al. (2007), and Foroughi & Aouni (2012), which consider respectively the following sets of weights:

$$S_{\mathrm{W}} = \{\boldsymbol{w} \in \mathbb{R}^m \mid w_1 \geq 2w_2 \geq \cdots \geq mw_m \geq 0,\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{w} = 1\},$$

$$S^{\varepsilon}_{\mathrm{W}} = \{\boldsymbol{w} \in \mathbb{R}^m \mid w_1 \geq 2w_2 \geq \cdots \geq mw_m,\ w_m \geq \varepsilon,\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{w} = 1\},$$

where $\varepsilon \geq 0$. Obviously, the set $S^{\varepsilon}_{\mathrm{W}}$ coincides with S_{W} when $\varepsilon = 0$. In both cases, the matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 0 & \dots & 0 & 0 \\ 0 & 2 & -3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & m-1 & -m \\ 0 & 0 & 0 & \dots & 0 & m \end{pmatrix}, \tag{3}$$

the inverse matrix $\boldsymbol{A}^{-1}$ is

$$\boldsymbol{A}^{-1} = \boldsymbol{H} = (\boldsymbol{h}_1, \boldsymbol{h}_2, \dots, \boldsymbol{h}_m) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1/2 & 1/2 & \dots & 1/2 \\ 0 & 0 & 1/3 & \dots & 1/3 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1/m \end{pmatrix}, \tag{4}$$

and normalizing the columns of $\boldsymbol{A}^{-1}$ gives the matrix

$$\boldsymbol{K} = (\boldsymbol{k}_1, \boldsymbol{k}_2, \dots, \boldsymbol{k}_m) = \begin{pmatrix} 1 & 2/3 & 6/11 & \dots & 1/\sum_{j=1}^m 1/j \\ 0 & 1/3 & 3/11 & \dots & 1/\sum_{j=1}^m 2/j \\ 0 & 0 & 2/11 & \dots & 1/\sum_{j=1}^m 3/j \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1/\sum_{j=1}^m m/j \end{pmatrix}. \tag{5}$$

3. Wang, Chin and Yang's models, and the solutions given by Ahn

As mentioned in the introduction, several models have been proposed within the framework of mathematical programming in order to obtain a common scoring vector for all candidates (see, for instance, Wang et al. (2007), Foroughi & Aouni (2012), Ahn (2017a,b), and Contreras (2020) for a literature review of models that determine a common set of weights both in this field and in the DEA context). Two linear programming models that have received a great deal of attention have been those proposed by Wang et al. (2007). Both models have been thoroughly analyzed by Foroughi & Aouni (2012), who have identified the following shortcomings:

1. The models may have multiple optimal solutions that provide different rankings of the candidates.
2. The optimal scoring vectors are based on the results obtained by inefficient candidates, so the final scores of candidates are very sensitive to the results obtained by the worst candidates.
3. Some of the weights of the optimal solutions could be zero.

On the other hand, Ahn (2017a) has provided closed expressions for the optimal solutions of both models but, as we will see in the following subsections, such solutions are not valid in all cases.

3.1. Wang, Chin and Yang's first model

The first model proposed by Wang et al. (2007) is the following:

$$\begin{aligned}
&\text{Max } \ \alpha, \\
&\text{s.t. } \ Z_i = \sum_{j=1}^{m} v_{ij} w_j \geq \alpha, \quad i = 1, \ldots, n, \qquad \text{(LP-1)} \\
&\qquad \boldsymbol{w} \in S_{\mathrm{W}}.
\end{aligned}$$

Once the optimal solution of this model is obtained, all candidates are evaluated using the optimal scoring vector and ordered according to the number of points they have received. The optimal solution of (LP-1) was established by Ahn (2017a) in the following statement.

Claim 1. *The optimal solution to (LP-1) is* $\alpha^* = \max_{1\leq j\leq m}\{\underline{\alpha}_j\}$*, where* $\underline{\alpha}_j = \min_{1\leq i\leq n}\{\boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{k}_j\}$*,* $\boldsymbol{k}_j$ *is the jth column of* $\boldsymbol{K}$ *(where* $\boldsymbol{K}$ *is given by expression (5)),* $\boldsymbol{v}_i^{\mathrm{T}} = (v_{i1}, \ldots, v_{im})$*, and* $\boldsymbol{w}^* = \boldsymbol{k}_j$ *for each* $j \in \{1, \ldots, m\}$ *such that* $\underline{\alpha}_j = \alpha^*$*.*

To illustrate the result obtained, Ahn (2017a) uses an example proposed by Cook & Kress (1990) (that was also used by Wang et al. (2007)) where 20 voters rank 4 out of 6 candidates on a ballot (see Table 1).

Table 1: Ranks obtained by each candidate (Cook and Kress example).

Candidate	v_{i1}	v_{i2}	v_{i3}	v_{i4}
A	3	3	4	3
B	4	5	5	2
C	6	2	3	2
D	6	2	2	6
E	0	4	3	4
F	1	4	3	3

According to Claim 1, the optimal solution can be obtained as follows. First the matrix $\boldsymbol{VK}$ is calculated (see expression (6)), where $\boldsymbol{V}$ is the matrix containing the ranks obtained by the candidates. Then the minimum value of each column is chosen (they appear highlighted in expression (6)) and, among these, the maximum value is the optimal solution. Therefore, in this example $\alpha^* = \max\{0, 1.\overline{3}, 1.\overline{63}, 1.92\} = 1.92$, and $\boldsymbol{w}^* = \boldsymbol{k}_4 = (12/25, 6/25, 4/25, 3/25)$.

$$\boldsymbol{VK} = \begin{pmatrix} 3 & 3 & 4 & 3 \\ 4 & 5 & 5 & 2 \\ 6 & 2 & 3 & 2 \\ 6 & 2 & 2 & 6 \\ 0 & 4 & 3 & 4 \\ 1 & 4 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2/3 & 6/11 & 12/25 \\ 0 & 1/3 & 3/11 & 6/25 \\ 0 & 0 & 2/11 & 4/25 \\ 0 & 0 & 0 & 3/25 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 3.\overline{18} & 3.16 \\ 4 & 4.\overline{3} & 4.\overline{45} & 4.16 \\ 6 & 4.\overline{6} & 4.\overline{36} & 4.08 \\ 6 & 4.\overline{6} & 4.\overline{18} & 4.4 \\ \boxed{0} & \boxed{1.\overline{3}} & \boxed{1.\overline{63}} & \boxed{1.92} \\ 1 & 2 & 2.\overline{18} & 2.28 \end{pmatrix}. \tag{6}$$

Notice that the element ij of the matrix $\boldsymbol{VK}$ is the score obtained by candidate A_i when is evaluated using the scoring vector $\boldsymbol{k}_j$. Hence, the fourth column of the matrix $\boldsymbol{VK}$ gives the candidate scores for the optimal scoring vector. These scores coincide with those obtained by Wang et al. (2007) through MS-Excel Solver. So, in this example Claim 1 provides the correct solution. This fact is due to the following reason. If, for candidate E, we consider maximizing her score in S_{W}; that is, the following linear programming model,

$$\begin{aligned} &\text{Max } \ 4w_2 + 3w_3 + 4w_4, \\ &\quad \text{s.t. } \ \boldsymbol{w} \in S_{\mathrm{W}}, \end{aligned}$$

the solution is obtained at an extreme point, in this case at the extreme point $\boldsymbol{k}_4$. As for that scoring vector candidate E is the one that obtains the lowest score, that score is the value of α^* and $\boldsymbol{k}_4$ the optimal scoring vector. In fact, using a similar argument, we can state the following theorem.

Theorem 2. *Consider (LP-1) and let $\mathbf{Z} = (Z_{ij}) = \boldsymbol{VK}$ where $\boldsymbol{V}$ is the matrix containing the ranks obtained by the candidates and $\boldsymbol{K}$ is the matrix given by expression (5). If there exists an element Z_{ij} of $\mathbf{Z}$ such that $Z_{ij} = \max_{1\le k\le m}\{Z_{ik}\}$ and $Z_{ij} = \min_{1\le k\le n}\{Z_{kj}\}$ then $\alpha^* = Z_{ij}$, $\boldsymbol{w}^* = \boldsymbol{k}_j$, and the jth column of $\mathbf{Z}$ provides the candidate scores for the optimal scoring vector.*

Proof. Suppose there exists an element Z_{ij} of the matrix $\mathbf{Z} = \boldsymbol{VK}$ such that $Z_{ij} = \max_{1\le k\le m}\{Z_{ik}\}$ and $Z_{ij} = \min_{1\le k\le n}\{Z_{kj}\}$. Consider the following linear programming model:

$$\begin{aligned} &\text{Max } \ Z_i = \sum_{j=1}^{m} v_{ij} w_j, \\ &\quad \text{s.t. } \ \boldsymbol{w} \in S_{\mathrm{W}}. \end{aligned} \tag{7}$$

It is well known that the solution of this linear programming model is obtained at an extreme point of S_{W}. Therefore, according to Corollary 1 and given that $Z_{ij} = \max_{1\le k\le m}\{Z_{ik}\}$, the extreme point $\boldsymbol{k}_j$ is a solution of model (7). Since $Z_{ij} = \min_{1\le k\le n}\{Z_{kj}\}$, then $\boldsymbol{w}^* = \boldsymbol{k}_j$ is a solution of (LP-1), and $\alpha^* = Z_{ij}$. □

For instance, the condition given in Theorem 2 is also fulfilled if we consider the following sets of candidates: {A, B, C, D, F}, and {A, B, C, D}. Nevertheless, it is important to note that if the condition is not fulfilled, the thesis of Claim 1 may not necessarily be true. For example, if we only consider candidates B, C, and D, we have

$$\boldsymbol{VK} = \begin{pmatrix} 4 & 5 & 5 & 2 \\ 6 & 2 & 3 & 2 \\ 6 & 2 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2/3 & 6/11 & 12/25 \\ 0 & 1/3 & 3/11 & 6/25 \\ 0 & 0 & 2/11 & 4/25 \\ 0 & 0 & 0 & 3/25 \end{pmatrix} = \begin{pmatrix} \boxed{4} & \boxed{4.\overline{3}} & 4.\overline{45} & 4.16 \\ 6 & 4.\overline{6} & 4.\overline{36} & \boxed{4.08} \\ 6 & 4.\overline{6} & \boxed{4.\overline{18}} & 4.4 \end{pmatrix}.$$

According to Claim 1, we get $\alpha^* = \max\{4, 4.\overline{3}, 4.\overline{18}, 4.08\} = 4.\overline{3}$, and $\boldsymbol{w}^* = \boldsymbol{k}_2 = (2/3, 1/3, 0, 0)$. However, the LINGO software package returns the following solution of (LP-1) (only considering candidates B, C, and D): $\alpha^* = 4.4$, and $\boldsymbol{w}^* = (0.6, 0.3, 0.1, 0)$. Hence, the optimal solution given in Claim 1 is not valid in all cases. The justification for this fact is that, in general, the columns of the matrix $\boldsymbol{K}$ do not correspond to the vertices of the feasible set of (LP-1) because, in addition to the constraints of the set S_{W}, it is also necessary to consider the constraints $Z_i \geq \alpha$, $i = 1, \ldots, n$. Another alternative argumentation can be provided taking into account that (LP-1) can be written as the following linear max-min programming:

$$\begin{aligned} &\text{Max} \quad \min_{i=1,\ldots,n} \left\{ \sum_{j=1}^{m} v_{ij} w_j \right\}, \\ &\quad \text{s.t.} \ \ \boldsymbol{w} \in S_{\text{W}}. \end{aligned}$$

It is known that if the maximum value is non-zero (what is true in this case), then the maximum is attained on the boundary of the feasible set (see Posner & Wu (1981)), but it cannot be ensured that it is reached at an extreme point of the feasible set (as Ahn (2017a) does).

It is worth making a final comment about (LP-1). If we take into account all candidates in the previous example, the final ranking is (see the fourth column of the matrix $\boldsymbol{VK}$ in expression (6))

$$\text{D} \succ \text{B} \succ \text{C} \succ \text{A} \succ \text{F} \succ \text{E}.$$

However, given that candidates A, E, and F can never be the winners because they are strictly dominated by B, we can rule out these candidates. In this case we have seen that the optimal scoring vector is $\boldsymbol{w}^* = (0.6, 0.3, 0.1, 0)$ and the final ranking of the candidates is

$$\text{C}(4.5) \succ \text{D}(4.4) \sim \text{B}(4.4),$$

that is, the winning candidate depends on whether candidates who cannot win are considered, which is a serious drawback from the point of view of Social Choice Theory (in this regard see Llamazares & Peña (2009) for an analysis of some models proposed in the literature).

3.2. Wang, Chin and Yang's second model

The second model proposed by Wang et al. (2007) is the following:

$$\begin{aligned} &\text{Max} \ \ \alpha, \\ &\quad \text{s.t.} \ \ \alpha \leq Z_i = \sum_{j=1}^{m} v_{ij} w_j \leq 1, \quad i = 1, \ldots, n, \\ &\qquad\quad 1 \geq w_1 \geq 2w_2 \geq \cdots \geq m w_m \geq 0. \end{aligned} \tag{LP-2}$$

For this model Ahn (2017a) establishes the following statement.

Claim 2. *The optimal solution to (LP-2) is obtained by* $\alpha^* = \min\{(1/\beta)\boldsymbol{V}\boldsymbol{h}_m\}$*, and the optimal weight vector is* $\boldsymbol{w}^* = (1/\beta)\boldsymbol{h}_m$*, where* $\beta = \max\{\boldsymbol{V}\boldsymbol{h}_m\}$.[2]

According to Claim 2, the optimal scoring vector is always a vector proportional to $\boldsymbol{h}_m$. Ahn (2017a) illustrates the result through the example proposed by Cook and Kress (Table 1), which was also used by Wang et al. (2007). However, although in that example the solution provided by Claim 2 is the correct one (and the winner is candidate D), this is not always the case. For instance, if in that example we only consider candidates B, C, and D, we have

$$\boldsymbol{V}\boldsymbol{h}_m = \begin{pmatrix} 4 & 5 & 5 & 2 \\ 6 & 2 & 3 & 2 \\ 6 & 2 & 2 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 26/3 \\ 17/2 \\ 55/6 \end{pmatrix}.$$

In accord with Claim 2, the optimal solution should be $\beta = 55/6$, $\boldsymbol{w}^* = (6/55, 3/55, 2/55, 3/110)$, and $\alpha^* = 51/55$, so that D $\succ$ B $\succ$ C. However, with the LINGO software package we find that the optimal solution of (LP-2) is $\alpha^* = 1$, and $\boldsymbol{w}^* = (8/63, 4/63, 2/63, 1/126)$. Therefore, all three candidates get the highest score, 1, and D $\sim$ B $\sim$ C. Notice also that, as in the first model, the winning candidates depends on whether candidates who cannot win are taken into account.

To understand some of the erroneous reasoning used in the proof of Claim 2 we are going to use an example, which is similar to the one proposed by Cook & Kress (1990) but with a slight modification. Suppose that, in the example proposed by Cook & Kress (1990), candidate E loses two fourth positions in favor of candidate D (see Table 2).

In this case, the matrix $\boldsymbol{V}\boldsymbol{H}$ is the following:

$$\boldsymbol{V}\boldsymbol{H} = \begin{pmatrix} 3 & 3 & 4 & 3 \\ 4 & 5 & 5 & 2 \\ 6 & 2 & 3 & 2 \\ 6 & 2 & 2 & 8 \\ 0 & 4 & 3 & 2 \\ 1 & 4 & 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/2 & 1/2 & 1/2 \\ 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} = \begin{pmatrix} 3 & 9/2 & 35/6 & 79/12 \\ 4 & 13/2 & 49/6 & 26/3 \\ 6 & 7 & 8 & 17/2 \\ 6 & 7 & 23/3 & 29/3 \\ 0 & 2 & 3 & 7/2 \\ 1 & 3 & 4 & 19/4 \end{pmatrix}. \qquad (8)$$

When we divide the elements of each column by the maximum of that column we get the matrix

[2] We recall that $\boldsymbol{h}_m$ is the last column of the matrix $\boldsymbol{A}^{-1} = \boldsymbol{H}$ given in expression (4).

Table 2: Ranks obtained by each candidate (modified Cook and Kress example).

Candidate	v_{i1}	v_{i2}	v_{i3}	v_{i4}
A	3	3	4	3
B	4	5	5	2
C	6	2	3	2
D	6	2	2	8
E	0	4	3	2
F	1	4	3	3

$$\begin{pmatrix} 1/2 & 9/14 & 5/7 & 79/116 \\ 2/3 & 13/14 & 1 & 26/29 \\ 1 & 1 & 48/49 & 51/58 \\ 1 & 1 & 46/49 & 1 \\ 0 & 2/7 & 18/49 & 21/58 \\ 1/6 & 3/7 & 24/49 & 57/116 \end{pmatrix} \approx \begin{pmatrix} 0.5 & 0.6429 & 0.7143 & 0.6810 \\ 0.6667 & 0.9286 & 1 & 0.8966 \\ 1 & 1 & 0.9796 & 0.8793 \\ 1 & 1 & 0.9388 & 1 \\ \boxed{0} & \boxed{0.2857} & \boxed{0.3673} & \boxed{0.3621} \\ 0.1667 & 0.4286 & 0.4898 & 0.4914 \end{pmatrix},$$

where we have highlighted the minimum value of each column. Ahn (2017a) argues that the maximum of these minimum values is always obtained in the last column ($\boldsymbol{h}_m$), which, as can be seen in this example, is not true (in this example α^* would be $18/49 \approx 0.3673$ and not $21/58 \approx 0.3621$). Moreover, the optimal solution does not have to be reached in a scoring vector proportional to one of the columns of matrix $\boldsymbol{H}$. The optimal solution returned by the LINGO software package is $\alpha^* = 0.38$, $\boldsymbol{w}^* = (0.12, 0.06, 0.04, 0.01)$, and

$$\text{B}(1) \sim \text{D}(1) \succ \text{C}(0.98) \succ \text{A}(0.73) \succ \text{F}(0.51) \succ \text{E}(0.38).$$

Note also that in this example there is an element of the matrix that is the maximum of its row and the minimum of its column (the $(5, 3)$ entry of the matrix). However, the column to which it belongs is not the optimal scoring vector for the model; that is, it cannot be established a result similar to that of Theorem 2.

4. Foroughi and Aouni's models

In order to avoid the shortcomings that Foroughi & Aouni (2012) had encountered in the models proposed by Wang et al. (2007), these authors suggest the following model:

$$\begin{aligned} &\text{Min } \beta, \\ &\text{s.t. } Z_i = \sum_{j=1}^{m} v_{ij} w_j \leq \beta, \quad i = 1, \ldots, n, \\ &\quad \boldsymbol{w} \in S^{\varepsilon}_{\mathrm{W}}. \end{aligned} \tag{LP-3}$$

As can be observed in (LP-3), these authors consider a minimax model so that the optimal solution depends only on efficient candidates. In addition, they introduce the constraint $w_m \geq \varepsilon$ to ensure that the weights are non-zero when $\varepsilon > 0$. However, as the same authors point out, this model may have multiple optimal solutions that provide different rankings of the candidates. To find an example where this problem arises, we can establish a result similar to the one given in Theorem 2 for the following general model,

$$\begin{aligned} &\text{Min } \beta, \\ &\text{s.t. } Z_i = \sum_{j=1}^{m} v_{ij} w_j \leq \beta, \quad i = 1, \ldots, n, \\ &\quad \boldsymbol{w} \in S_{(\boldsymbol{A},\boldsymbol{\lambda})}, \end{aligned} \tag{LP-4}$$

which encompasses model (LP-3).

Theorem 3. *Consider (LP-4) and let $\boldsymbol{Z} = (Z_{ij}) = \boldsymbol{V}\boldsymbol{A}^{-1}\boldsymbol{D}$ where $\boldsymbol{V}$ is the matrix containing the ranks obtained by the candidates, and $\boldsymbol{D}$ is the matrix given by expression (2). If there exists an element Z_{ij} of $\boldsymbol{Z}$ such that $Z_{ij} = \min_{1\leq k\leq m}\{Z_{ik}\}$ and $Z_{ij} = \max_{1\leq k\leq n}\{Z_{kj}\}$, then an optimal scoring vector is the jth column of $\boldsymbol{A}^{-1}\boldsymbol{D}$, and the jth column of $\boldsymbol{Z}$ provides the candidate scores for this optimal scoring vector.*

Proof. Suppose there exists an element Z_{ij} of $\boldsymbol{Z} = \boldsymbol{V}\boldsymbol{A}^{-1}\boldsymbol{D}$ such that $Z_{ij} = \min_{1\leq k\leq m}\{Z_{ik}\}$ and $Z_{ij} = \max_{1\leq k\leq n}\{Z_{kj}\}$. Consider the following linear programming model:

$$\begin{aligned} &\text{Min } Z_i = \sum_{j=1}^{m} v_{ij} w_j, \\ &\text{s.t. } \boldsymbol{w} \in S_{(\boldsymbol{A},\boldsymbol{\lambda})}. \end{aligned} \tag{9}$$

It is well known that the solution of this linear programming model is obtained at an extreme point of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$. Therefore, according to Theorem 1 and given that $Z_{ij} = \min_{1\leq k\leq m}\{Z_{ik}\}$, the jth column of $\boldsymbol{A}^{-1}\boldsymbol{D}$ is a

Table 3: Ranks obtained by each candidate.

Candidate	v_{i1}	v_{i2}	v_{i3}
A	8	5	7
B	7	7	5
C	3	6	4
D	1	1	1
E	6	7	10
F	5	4	3

solution of model (9). Since $Z_{ij} = \max_{1\leq k\leq n}\{Z_{kj}\}$, then the jth column of $\boldsymbol{A}^{-1}\boldsymbol{D}$ is a solution of (LP-4), and $\beta^* = Z_{ij}$. □

According to Theorem 3, there are multiple optimal solutions if there is a row in matrix $\boldsymbol{Z}$ where several elements are the minimum of that row and those same elements are the maximums in their respective columns. In this case, those columns, which are extreme points of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$, are optimal solutions of model (LP-4). For instance, consider that 30 voters rank 3 out of 6 candidates on a ballot and the ranks obtained by each candidate are shown in Table 3. In the case of model (LP-3) and given $\varepsilon \geq 0$, $\boldsymbol{A}$ is the matrix of order 3 given by expression (3), and $\boldsymbol{\lambda} = (0, 0, 3\varepsilon)$. Therefore, the extreme points of $S^{\varepsilon}_{\mathrm{W}}$ are the columns of the matrix

$$\boldsymbol{A}^{-1}\boldsymbol{D} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} \frac{2-11\varepsilon}{2} & 0 & 0 \\ 0 & \frac{2-11\varepsilon}{3} & 0 \\ 3\varepsilon & 3\varepsilon & \frac{6}{11} \end{pmatrix} = \begin{pmatrix} \frac{2-5\varepsilon}{2} & \frac{2-2\varepsilon}{3} & \frac{6}{11} \\ \frac{3\varepsilon}{2} & \frac{1-\varepsilon}{3} & \frac{3}{11} \\ \varepsilon & \varepsilon & \frac{2}{11} \end{pmatrix},$$

and the matrix $\boldsymbol{Z} = \boldsymbol{V}\boldsymbol{A}^{-1}\boldsymbol{D}$ is given by

$$\boldsymbol{Z} = \begin{pmatrix} 8 & 5 & 7 \\ 7 & 7 & 5 \\ 3 & 6 & 4 \\ 1 & 1 & 1 \\ 6 & 7 & 10 \\ 5 & 4 & 3 \end{pmatrix} \begin{pmatrix} \frac{2-5\varepsilon}{2} & \frac{2-2\varepsilon}{3} & \frac{6}{11} \\ \frac{3\varepsilon}{2} & \frac{1-\varepsilon}{3} & \frac{3}{11} \\ \varepsilon & \varepsilon & \frac{2}{11} \end{pmatrix} = \begin{pmatrix} \frac{16-11\varepsilon}{2} & 7 & 7 \\ 7-2\varepsilon & 7-2\varepsilon & \frac{73}{11} \\ \frac{11\varepsilon+6}{2} & 4 & 4 \\ 1 & 1 & 1 \\ \frac{11\varepsilon+12}{2} & \frac{11\varepsilon+19}{3} & 7 \\ \frac{10-7\varepsilon}{2} & \frac{14-5\varepsilon}{3} & \frac{48}{11} \end{pmatrix}.$$

By Remark 1, $S^{\varepsilon}_{\mathrm{W}}$ is not empty if and only if $11\varepsilon/2 \leq 1$, or equivalently, when $\varepsilon \leq 2/11$. It is easy to check that when $\varepsilon \in [0, 2/11]$ the entries $(1,2)$ and $(1,3)$ of matrix $\boldsymbol{Z}$ are the minimum of row 1 and the maximum of columns 2 and 3, respectively.[3] Therefore, the extreme points $((2-2\varepsilon)/3, (1-\varepsilon)/3, \varepsilon)$ and $(6/11, 3/11, 2/11)$ are both optimal solutions of model (LP-3). With the first scoring vector, the set of winners is

1. $\{\mathrm{A}, \mathrm{B}\}$ when $\varepsilon = 0$,
2. $\{\mathrm{A}\}$ when $0 < \varepsilon < 2/11$,
3. $\{\mathrm{A}, \mathrm{E}\}$ when $\varepsilon = 2/11$,

while with the second scoring vector, $(6/11, 3/11, 2/11)$, the set of winners is always $\{\mathrm{A}, \mathrm{E}\}$ for any $\varepsilon \in [0, 2/11]$. So, for any $\varepsilon < 2/11$, the winning candidates depend on the scoring vector chosen as optimal solution.

It is worth noting that when $\varepsilon = 0$, the LINGO software returns the optimal solution $(2/3, 1/3, 0)$, and candidates A and B are the winners with this scoring vector. Analogously, when $\varepsilon = 1/11$, the solution provided by the LINGO software is $(20/33, 10/33, 1/11)$ (again the second column of the matrix $\boldsymbol{A}^{-1}\boldsymbol{D}$), and the winner is A. However, in both cases, candidate E could also be a winner if the vector $(6/11, 3/11, 2/11)$ was used instead of $(2/3, 1/3, 0)$ or $(20/33, 10/33, 1/11)$.

Foroughi & Aouni (2012) also suggest a more general model than (LP-3) in order to avoid multiple solutions and increase the discriminatory power of the model, making it more likely to select a single winner. With these purposes in mind, the model these authors propose is the following:

$$\begin{aligned} \text{Min} \quad & \theta\beta - (1-\theta)\alpha, \\ \text{s.t.} \quad & \alpha \leq \sum_{j=1}^{m} v_{ij} w_j \leq \beta, \quad i = 1, \ldots, n, \\ & \boldsymbol{w} \in S^{\varepsilon}_{\mathrm{W}}, \end{aligned} \tag{LP-5}$$

where $\theta \in [0, 1]$ and is determined by the decision maker. Nevertheless, we can use the previous example (Table 3) to show that this model can also have multiple solutions. For instance, if we consider $\varepsilon = 0$ and $\theta = 0.5$, the LINGO software returns the optimal solution $(6/11, 3/11, 2/11)$ (and candidates A and E are the winners), but it is easy to check that $(2/3, 1/3, 0)$ is also an optimal solution, and with this scoring vector,

[3]When $\varepsilon = 2/11$, $S^{\varepsilon}_{\mathrm{W}}$ has only one element, the vector $(6/11, 3/11, 2/11)$, and the three columns of the matrix $\boldsymbol{A}^{-1}\boldsymbol{D}$ are equal to $(6/11, 3/11, 2/11)^{\mathrm{T}}$ (see the last comment of Remark 2).

the winning candidates are A and B. Analogously, if we take $\varepsilon = 1/11$ and $\theta = 0.5$, the LINGO software returns $(6/11, 3/11, 2/11)$ again, but $(2/3, 1/3, 0)$ remains an optimal solution as well.

5. Surrogate and centroid weights

As we have seen in the previous sections, the choice of a scoring vector by means of a mathematical programming model does not seem very appropriate unless we can guarantee the uniqueness of the solution. An alternative approach is the use of surrogate weights, which are often used in multiattribute decision making problems when only ordinal information about the importance of the attributes is available (usually, an ordinal ranking of the attributes). In this context of surrogate weights, several sets of weights have been proposed in the literature[4] (see also Roszkowska (2013), and Danielson & Ekenberg (2014)):

1. Equal weights (EW):

$$w_j = \frac{1}{m}, \quad j = 1, \ldots, m.$$

2. Rank sum (RS) weights (Stillwell et al. (1981)):

$$w_j = \frac{m+1-j}{\sum_{k=1}^{m}(m+1-k)} = \frac{2(m+1-j)}{m(m+1)}, \quad j = 1, \ldots, m.$$

   It is important to note that the procedure obtained by using rank sum weights corresponds to a well-known method for aggregating rankings, the normalized truncated Borda rule (see, for instance, Fishburn (1974)).

3. Rank reciprocal (RR) weights (Stillwell et al. (1981)):

$$w_j = \frac{1/j}{\sum_{k=1}^{m} 1/k}, \quad j = 1, \ldots, m.$$

4. Rank exponent (RE) weights (Stillwell et al. (1981)):

$$w_j = \frac{(m+1-j)^p}{\sum_{k=1}^{m}(m+1-k)^p}, \quad j = 1, \ldots, m.$$

5. Rank order centroid (ROC) weights (Barron (1992)):

$$w_j = \frac{1}{m}\sum_{k=j}^{m}\frac{1}{k}, \quad j = 1, \ldots, m.$$

[4] For the sake of simplicity, and without loss of generality, we can assume that $w_1 \geq w_2 \geq \cdots \geq w_m$.

6. Combined sum and reciprocal weights (Danielson & Ekenberg (2014)):

$$w_j = \frac{\frac{1}{j} + \frac{m+1-j}{m}}{\sum_{k=1}^{m}\left(\frac{1}{k} + \frac{m+1-k}{m}\right)}, \quad j = 1, \dots, m.$$

7. Geometric weights (Danielson & Ekenberg (2014)):

$$w_j = \frac{s^{j-1}}{\sum_{k=1}^{m} s^{k-1}}, \quad j = 1, \dots, m,$$

where $s \in (0, 1)$.

It is well known (Barron (1992)) that the ROC vector is the centroid of the simplex

$$S = \{\boldsymbol{w} \in \mathbb{R}^m \mid w_1 \geq w_2 \geq \cdots \geq w_m \geq 0,\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{w} = 1\}, \tag{10}$$

and it is obtained as the arithmetic mean of the extreme points of S. For this reason, the ROC weights perform better than other surrogate weights when comparing the evaluations obtained by these weights with those that would be obtained using random vectors from S, which are called 'true weights' (see Kunsch & Ishizaka (2019)).[5]

As Kunsch & Ishizaka (2019) have shown, a similar result can be obtained for the centroid of any given simplex, because "its evaluation has on average the least squared deviation to all 'true weights' evaluations" (Kunsch & Ishizaka (2019)). Hence, given a simplex which represents the feasible set of weights, its centroid seems to be a good option when we have to choose a scoring vector.

Notice that in our context the centroid is calculated as the arithmetic mean of the extreme points of the simplex (see Ahn (2017b)). In fact, using a different methodology than that used in this paper, Ahn (2017b) shows the centroid of some simplices that commonly appear in the literature (the cases (**1a**) and (**4a**), which appear later in subsections 5.1 and 5.2, have been explicitly provided by Ahn (2017b), while cases (**1b**), (**1c**), and (**1d**) from subsection 5.1 can be obtained from a result shown in the same paper). However, the advantage of the approach used in this paper is that it allows obtaining a generic expression of the centroid for any simplex of the form $S_{(\boldsymbol{A},\boldsymbol{\lambda})} = \{\boldsymbol{w} \in \mathbb{R}^m \mid \boldsymbol{w} \geq \boldsymbol{0}, \boldsymbol{A}\boldsymbol{w} \geq \boldsymbol{\lambda},\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{w} = 1\}$, which encompasses the cases studied by Ahn (2017b).

By Theorem 1, the extreme points of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ are the columns of the matrix $\boldsymbol{A}^{-1}\boldsymbol{D}$ and, consequently the

[5]This conclusion had been reached in earlier simulation studies (see, for instance, Barron & Barrett (1996a,b), Ahn (2011)).

centroid of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ can be obtained as $m^{-1}\boldsymbol{A}^{-1}\boldsymbol{De}$.[6] By Remark 2 we get

$$\boldsymbol{De} = \begin{pmatrix} \lambda_1 + \mu_1^{-1}\delta & \lambda_1 & \dots & \lambda_1 \\ \lambda_2 & \lambda_2 + \mu_2^{-1}\delta & \dots & \lambda_2 \\ \dots & \dots & \dots & \dots \\ \lambda_m & \lambda_m & \dots & \lambda_m + \mu_m^{-1}\delta \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} m\lambda_1 + \mu_1^{-1}\delta \\ m\lambda_2 + \mu_2^{-1}\delta \\ \vdots \\ m\lambda_m + \mu_m^{-1}\delta \end{pmatrix},$$

where μ_k $(k = 1,\dots,m)$ is the sum of the elements of the kth column of $\boldsymbol{A}^{-1} = \boldsymbol{H} = (h_{ij})$ and $\delta = 1 - \sum_{k=1}^m \mu_k\lambda_k$. Therefore, the jth component of the centroid of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ is

$$w_j = \frac{1}{m}\sum_{k=1}^m h_{jk}(m\lambda_k + \mu_k^{-1}\delta) = \sum_{k=1}^m h_{jk}\lambda_k + \frac{1}{m}\delta\sum_{k=1}^m h_{jk}\mu_k^{-1}. \tag{11}$$

Typical examples of $\boldsymbol{\lambda}$ that have been used in the literature are the following (see, for instance, Cook & Kress (1990), Wang et al. (2007) and Foroughi & Aouni (2012)):

(**a**) $\lambda_k = 0$ $(k = 1,\dots,m)$. In this case $\delta = 1$, and expression (11) becomes

$$w_j = \frac{1}{m}\sum_{k=1}^m h_{jk}\mu_k^{-1}.$$

(**b**) $\lambda_k = 0$ $(k = 1,\dots,m-1)$, and $\lambda_m = \varepsilon$. In this case $\delta = 1 - \varepsilon\mu_m$, and expression (11) becomes

$$w_j = \varepsilon h_{jm} + \frac{1}{m}(1 - \varepsilon\mu_m)\sum_{k=1}^m h_{jk}\mu_k^{-1}.$$

Note that the maximum value of ε,[7] which is obtained when $\delta = 0$, is $\varepsilon^* = \mu_m^{-1}$ and, in this case,

$$w_j = \mu_m^{-1}h_{jm};$$

that is, the only element of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ is the last column of $\boldsymbol{A}^{-1}$ normalized to sum 1.

(**c**) $\lambda_k = \varepsilon$ $(k = 1,\dots,m)$. In this case $\delta = 1 - \varepsilon\sum_{k=1}^m \mu_k$, and expression (11) becomes

$$w_j = \varepsilon\sum_{k=1}^m h_{jk} + \frac{1}{m}\left(1 - \varepsilon\sum_{k=1}^m \mu_k\right)\sum_{k=1}^m h_{jk}\mu_k^{-1}.$$

The maximum value of ε (obtained with $\delta = 0$) is $\varepsilon^* = \left(\sum_{k=1}^m \mu_k\right)^{-1}$ and, in this case,

$$w_j = \left(\sum_{k=1}^m \mu_k\right)^{-1}\sum_{k=1}^m h_{jk};$$

that is, the only element of $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ is the vector $\boldsymbol{A}^{-1}\boldsymbol{e}$ normalized to sum 1.

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[6] Notice that by Remark 2 we can use also the expression $m^{-1}\boldsymbol{A}^{-1}\boldsymbol{De}$ when the simplex $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ has only one vector.

[7] In this case as well as in the following two cases, (**c**) and (**d**), as ε increases, the set $S_{(\boldsymbol{A},\boldsymbol{\lambda})}$ decreases until it collapses into a single vector, which is the one obtained with ε^*.

(**d**) $\lambda_k = \varepsilon/k$ $(k = 1, \dots, m)$. In this case $\delta = 1 - \varepsilon \sum_{k=1}^{m} \frac{\mu_k}{k}$, and expression (11) becomes

$$w_j = \varepsilon \sum_{k=1}^{m} \frac{h_{jk}}{k} + \frac{1}{m}\left(1 - \varepsilon \sum_{k=1}^{m} \frac{\mu_k}{k}\right) \sum_{k=1}^{m} h_{jk}\mu_k^{-1}.$$

The maximum value of ε (obtained when $\delta = 0$) is $\varepsilon^* = \left(\sum_{k=1}^{m} \frac{\mu_k}{k}\right)^{-1}$ and, in this case,

$$w_j = \left(\sum_{k=1}^{m} \frac{\mu_k}{k}\right)^{-1} \sum_{k=1}^{m} \frac{h_{jk}}{k};$$

that is, the only element of $S_{(\boldsymbol{A},\lambda)}$ is the vector $\boldsymbol{A}^{-1}(1, 1/2, \dots, 1/m)^{\mathrm{T}}$ normalized to sum 1.

Although it is obvious, it is worth noting that case (**a**) can be obtained as a particular case of the others by taking $\varepsilon = 0$. When it is desired for ε to be different from zero, and given that this value may determine the final ranking of the candidates, the question that naturally arises is: what is the most suitable value for this parameter? A solution that has been used in the literature (in models similar to those considered in this paper) is to choose the maximum possible value of ε (Cook & Kress (1990)), although in some cases it does not seem to be the best choice (Llamazares (2016)). Another possibility that has been proposed in the literature is to consider for each candidate the average value of the scores obtained when ε varies in its interval of values (Llamazares & Peña (2013)). Notice that in cases (**b**), (**c**) and (**d**), the jth component of the scoring vector can be expressed as $w_j = p_j + q_j\,\varepsilon$. Therefore, the score obtained by candidate A_o when evaluated with the vector with weights w_j is

$$Z_o(\varepsilon) = \sum_{j=1}^{m} v_{oj} w_j = \sum_{j=1}^{m} (v_{oj} p_j + v_{oj} q_j\,\varepsilon),$$

and the average value will be

$$\overline{Z}_o = \frac{1}{\varepsilon^*} \int_0^{\varepsilon^*} Z_o(\varepsilon)\, d\varepsilon = \frac{1}{\varepsilon^*} \sum_{j=1}^{m} \int_0^{\varepsilon^*} (v_{oj} p_j + v_{oj} q_j\,\varepsilon)\, d\varepsilon = \sum_{j=1}^{m} \left(v_{oj} p_j + v_{oj} q_j \frac{\varepsilon^*}{2}\right)$$
$$= \sum_{j=1}^{m} v_{oj} \left(p_j + q_j \frac{\varepsilon^*}{2}\right),$$

that is, the average value coincides with the value of the function at the point $\varepsilon^*/2$. [8]

Notice that the weights obtained with the value $\varepsilon = \varepsilon^*/2$ correspond to the average of the weights obtained with $\varepsilon = 0$ and $\varepsilon = \varepsilon^*$; that is,

$$p_j + q_j \frac{\varepsilon^*}{2} = \frac{1}{2} p_j + \frac{1}{2}(p_j + q_j\,\varepsilon^*).$$

[8] This result is not surprising since $Z_o(\varepsilon)$ is a linear function of ε, so its average value in the interval $[0, \varepsilon^*]$ is obtained by evaluating the function at the midpoint of that interval.

Therefore, calculating the average value of the function $Z_o(\varepsilon)$ when ε varies in the interval $[0, \varepsilon^*]$ is equivalent to using the weights resulting from averaging the weights obtained for $\varepsilon = 0$ and those obtained for $\varepsilon = \varepsilon^*$.[9]

Now we focus our attention on matrix $\boldsymbol{A}$, and in the following two subsections we are going to consider two generic cases which encompass the most commonly used sets of constraints in the literature. In the third subsection we will show some weighting vectors that can be obtained from the expressions provided in the first two subsections.

5.1. First generic matrix

The first matrix is given by

$$\boldsymbol{A} = \begin{pmatrix} \beta_1 & -\beta_2 & \dots & 0 & 0 \\ 0 & \beta_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta_{m-1} & -\beta_m \\ 0 & 0 & \dots & 0 & \beta_m \end{pmatrix},$$

where $0 < \beta_1 \leq \beta_2 \leq \dots \leq \beta_m$.[10] The constraints $\boldsymbol{Aw} \geq \boldsymbol{0}$ are

$$\beta_1 w_1 \geq \beta_2 w_2 \geq \beta_3 w_3 \geq \dots \geq \beta_m w_m \geq 0,$$

and, either in this form or in the more general one of $\boldsymbol{Aw} \geq \boldsymbol{\lambda}$, they have frequently appeared in the literature. For instance,

1. When $\beta_j = 1, \ (j = 1, \dots, m)$, they are the constraints introduced by Cook & Kress (1990) in their pioneering model and also the ones that allow obtaining the ROC weights as the centroid of a simplex (see expression (10)).
2. When $\beta_j = j, \ (j = 1, \dots, m)$, they are the constraints introduced by Noguchi et al. (2002), which have subsequently been used in other models (for example, those analyzed in sections 3 and 4 of this paper).

[9]The idea of using a combination of known weights has been previously used in the literature (see, for instance, the combined sum and reciprocal weights proposed by Danielson & Ekenberg (2014)), but without providing a clear justification for its use.

[10]These conditions are necessary to ensure that $w_j \geq w_{j+1}, \ (j = 1, \dots, m-1)$.

An equivalent way to represent these constraints is $w_j \geq \alpha_j w_{j+1},\ (j = 1, \ldots, m-1)$ (see, for instance, Ahn (2017b)); but the advantage of using the form $\beta_j w_j \geq \beta_{j+1} w_{j+1}$ is that it allows expressing matrix $\boldsymbol{A}^{-1}$ in a more compact way. It is easy to check that the inverse matrix of $\boldsymbol{A}$ is[11]

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 1/\beta_1 & 1/\beta_1 & \ldots & 1/\beta_1 & 1/\beta_1 \\ 0 & 1/\beta_2 & \ldots & 1/\beta_2 & 1/\beta_2 \\ \multicolumn{5}{c}{\dotfill} \\ 0 & 0 & \ldots & 1/\beta_{m-1} & 1/\beta_{m-1} \\ 0 & 0 & \ldots & 0 & 1/\beta_m \end{pmatrix},$$

and, consequently, $\mu_k = \sum_{r=1}^{k} \frac{1}{\beta_r}$ and

$$h_{jk} = \begin{cases} \dfrac{1}{\beta_j} & \text{if } k \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

The explicit values of the weights for the two specific cases discussed earlier and a new one that we will introduce are as follows:

(**1**) When $\beta_j = 1,\ (j = 1, \ldots, m)$, we have $\mu_k = k$ and

$$h_{jk} = \begin{cases} 1 & \text{if } k \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the centroid weights for the typical examples of $\boldsymbol{\lambda}$ seen previously (cases (**a**), (**b**), (**c**), and (**d**)) are as follows:

(1a) $w_j = \dfrac{1}{m} \sum_{k=j}^{m} \dfrac{1}{k}$, which are ROC weights.

(1b) $w_j = \varepsilon + \left(\dfrac{1}{m} - \varepsilon\right) \sum_{k=j}^{m} \dfrac{1}{k}$.

(1c) $w_j = (m + 1 - j)\varepsilon + \left(\dfrac{1}{m} - \dfrac{\varepsilon(m+1)}{2}\right) \sum_{k=j}^{m} \dfrac{1}{k}$.

(1d) $w_j = \dfrac{1}{m} \sum_{k=j}^{m} \dfrac{1}{k}$, which are again ROC weights. Note that although each λ_j depends on ε ($\lambda_j = \varepsilon / j$), the resulting weights do not depend on it.

[11]Explicit formulas for calculating inverse of triangular matrices have been given by Baliarsingh & Dutta (2015) and Baliarsingh et al. (2018).

Now, in cases (**b**), (**c**), and (**d**), we consider the maximum possible value of ε, ε^*. It is worth noting that in all three cases, surrogate weights known in the literature are obtained:

(**1b***) $w_j = \dfrac{1}{m} \quad \left(\varepsilon^* = \dfrac{1}{m}\right)$, which are equal weights.

(**1c***) $w_j = \dfrac{2(m+1-j)}{m(m+1)} \quad \left(\varepsilon^* = \dfrac{2}{m(m+1)}\right)$, which are RS weights.

(**1d***) $w_j = \dfrac{1}{m}\displaystyle\sum_{k=j}^{m}\dfrac{1}{k} \quad \left(\varepsilon^* = \dfrac{1}{m}\right)$, which are ROC weights.

Finally, other noteworthy weights for their previously mentioned interpretation are those obtained by averaging ROC weights (**1a**) with equal weights (**1b***), RS weights (**1c***), and ROC weights (**1d***).[12]

(**2**) When $\beta_j = j,\ (j = 1,\dots,m)$, we have $\mu_k = \sum_{r=1}^{k}\frac{1}{r}$ and

$$h_{jk} = \begin{cases} \dfrac{1}{j} & \text{if } k \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

To avoid an excessive proliferation of formulas, we will only show the expressions for the case (**a**), and the cases (**b**), (**c**), and (**d**) for ε^* (that is, the maximum possible value of ε):[13]

(**2a**) $w_j = \dfrac{1}{jm}\displaystyle\sum_{k=j}^{m}\left(\sum_{r=1}^{k}\dfrac{1}{r}\right)^{-1}.$

(**2b***) $w_j = \dfrac{1}{j}\left(\displaystyle\sum_{k=1}^{m}\dfrac{1}{k}\right)^{-1}$, which are RR weights.

(**2c***) $w_j = \dfrac{m+1-j}{j}\left(\displaystyle\sum_{k=1}^{m}\dfrac{m+1-k}{k}\right)^{-1}$, which, without normalization, are the weights

$$\left(m, \frac{m-1}{2}, \frac{m-2}{3}, \dots, \frac{3}{m-2}, \frac{2}{m-1}, \frac{1}{m}\right).$$

(**2d***) $w_j = \dfrac{1}{j}\left(\displaystyle\sum_{k=j}^{m}\dfrac{1}{k}\right)\left(\sum_{r=1}^{m}\dfrac{1}{r}\sum_{k=r}^{m}\dfrac{1}{k}\right)^{-1}.$

(**3**) We now consider the values $\beta_j = 1/(m+1-j),\ (j = 1,\dots,m)$. Notice that the constraints $\boldsymbol{A}\boldsymbol{w} \geq \boldsymbol{0}$, which are

$$\frac{1}{m}w_1 \geq \frac{1}{m-1}w_2 \geq \frac{1}{m-2}w_3 \geq \cdots \geq w_m \geq 0,$$

[12] Obviously, in this last case, we obtain ROC weights again.

[13] We recall here that an alternative way to calculate the weighting vectors obtained with ε^* is to normalize the following vectors to sum up to 1: the last column of $\boldsymbol{A}^{-1}$, the vector $\boldsymbol{A}^{-1}\boldsymbol{e}$, and the vector $\boldsymbol{A}^{-1}(1, 1/2, \dots, 1/m)^{\mathrm{T}}$.

are equivalent to

$$w_1 \geq \frac{m}{m-1} w_2 \geq \frac{m}{m-2} w_3 \geq \cdots \geq m w_m \geq 0,$$

which, as far as we know, have not been used in the literature. The values of μ_k and h_{jk} are $\mu_k = \sum_{r=1}^{k}(m+1-r) = \frac{k(2m+1-k)}{2}$ and

$$h_{jk} = \begin{cases} m+1-j & \text{if } k \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

and the weights for the cases (**a**), (**b***), (**c***), and (**d***) are as follows:

(3a) $w_j = \dfrac{2(m+1-j)}{m} \displaystyle\sum_{k=j}^{m} \frac{1}{k(2m+1-k)}.$

(3b*) $w_j = \dfrac{2(m+1-j)}{m(m+1)}$, which are RS weights.

(3c*) $w_j = (m+1-j)^2 \left(\dfrac{m(m+1)(2m+1)}{6} \right)^{-1}$, which are RE weights with $p = 2$.

(3d*) $w_j = \left(\dfrac{1}{2} \displaystyle\sum_{k=1}^{m} (2m+1-k) \right)^{-1} \displaystyle\sum_{k=j}^{m} \frac{m+1-j}{k} = \frac{4(m+1-j)}{3m+1} \frac{1}{m} \sum_{k=j}^{m} \frac{1}{k}$, which are ROC weights multiplied by $\frac{4(m+1-j)}{3m+1}$.

5.2. Second generic matrix

The second matrix we consider is given by

$$\boldsymbol{A} = \begin{pmatrix} \delta_1 & -\delta_1-\delta_2 & \delta_2 & \dots & 0 & 0 \\ 0 & \delta_2 & -\delta_2-\delta_3 & \dots & 0 & 0 \\ 0 & 0 & \delta_3 & \dots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & \delta_{m-1} & -\delta_{m-1}-\delta_m \\ 0 & 0 & 0 & \dots & 0 & \delta_m \end{pmatrix}$$

where $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_m$. The constraints $\boldsymbol{A}\boldsymbol{w} \geq \boldsymbol{0}$ are

$$\delta_1(w_1 - w_2) \geq \delta_2(w_2 - w_3) \geq \cdots \geq \delta_{m-1}(w_{m-1} - w_m) \geq \delta_m w_m \geq 0,$$

and, when $\delta_j = 1$, $(j = 1, \ldots, m)$, these restrictions have been used by several authors (see, among others, Stein et al. (1994), Contreras et al. (2005), Llamazares (2016), and Ahn (2017b)). The inverse matrix of $\boldsymbol{A}$

is

$$A^{-1} = \begin{pmatrix} \frac{1}{\delta_1} & \frac{1}{\delta_1}+\frac{1}{\delta_2} & \frac{1}{\delta_1}+\frac{1}{\delta_2}+\frac{1}{\delta_3} & \dots & \sum_{r=1}^{m-1}\frac{1}{\delta_r} & \sum_{r=1}^{m}\frac{1}{\delta_r} \\ 0 & \frac{1}{\delta_2} & \frac{1}{\delta_2}+\frac{1}{\delta_3} & \dots & \sum_{r=2}^{m-1}\frac{1}{\delta_r} & \sum_{r=2}^{m}\frac{1}{\delta_r} \\ 0 & 0 & \frac{1}{\delta_3} & \dots & \sum_{r=3}^{m-1}\frac{1}{\delta_r} & \sum_{r=3}^{m}\frac{1}{\delta_r} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{1}{\delta_{m-1}} & \frac{1}{\delta_{m-1}}+\frac{1}{\delta_m} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\delta_m} \end{pmatrix},$$

so $\mu_k = \sum_{r=1}^{k} \frac{r}{\delta_r}$ and

$$h_{jk} = \begin{cases} \sum_{r=j}^{k} \frac{1}{\delta_r} & \text{if } k \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we will show the explicit expressions for the weights in cases (**a**), (**b***), (**c***), and (**d***) when $\delta_j = 1,\ (j = 1, \dots, m)$,[14] in which case

$$h_{jk} = \begin{cases} k + 1 - j & \text{if } k \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu_k = \frac{k(k+1)}{2}$:

(4a) $w_j = \dfrac{2}{m} \sum_{k=j}^{m} \dfrac{k+1-j}{k(k+1)}.$

(4b*) $w_j = \dfrac{2(m+1-j)}{m(m+1)}$, which are the RS weights.

(4c*) $w_j = \left(\dfrac{1}{2}\sum_{k=1}^{m} k(k+1)\right)^{-1} \sum_{k=j}^{m}(k+1-j) = \dfrac{3(m+1-j)(m+2-j)}{m(m+1)(m+2)}.$

(4d*) $w_j = \left(\dfrac{1}{2}\sum_{k=1}^{m}(k+1)\right)^{-1} \sum_{k=j}^{m} \dfrac{k+1-j}{k} = \dfrac{4}{m(m+3)} \sum_{k=j}^{m} \dfrac{k+1-j}{k}.$

[14]We restrict our study to this case as it is the most common in the literature. However, similar to the first generic matrix, the weights could also be obtained when $\delta_j = j$ or when $\delta_j = 1/(m + 1 - j)$, $(j = 1, \dots, m)$.

5.3. Examples

Table 4 presents the weights obtained using the formulas of the previous subsections for the cases $m = 3$ and $m = 5$. As expected, the constraints imposed by both the matrix $\boldsymbol{A}$ and the vector $\boldsymbol{\lambda}$ are reflected in the weights, particularly in w_1. Thus, focusing on the matrix $\boldsymbol{A}$ and setting $\boldsymbol{\lambda} = \mathbf{0}$, the values of w_1, ordered from lowest to highest, correspond to the cases (**1a**), (**3a**), (**4a**), and (**2a**). Regarding the cases (*), when we fix the matrix $\boldsymbol{A}$, the values of w_1, ordered from lowest to highest, correspond to the cases (**b***), (**c***), and (**d***).

Table 4: Weighting vectors for $m = 3$ and $m = 5$.

Cases	Weights ($m = 3$)	Weights ($m = 5$)
ROC: (**1a**), (**1d***)	$\left(0.6\overline{1}, 0.2\overline{7}, 0.\overline{1}\right)$	$\left(0.45\overline{6}, 0.25\overline{6}, 0.15\overline{6}, 0.09, 0.04\right)$
EW: (**1b***)	$\left(0.\overline{3}, 0.\overline{3}, 0.\overline{3}\right)$	$\left(0.2, 0.2, 0.2, 0.2, 0.2\right)$
RS: (**1c***), (**3b***), (**4b***)	$\left(0.5, 0.\overline{3}, 0.1\overline{6}\right)$	$\left(0.\overline{3}, 0.2\overline{6}, 0.2, 0.1\overline{3}, 0.0\overline{6}\right)$
(**2a**)	$\left(0.\overline{73}, 0.\overline{20}, 0.\overline{06}\right)$	$\left(0.626, 0.213, 0.098, 0.046, 0.017\right)$
RR: (**2b***)	$\left(0.\overline{54}, 0.\overline{27}, 0.\overline{18}\right)$	$\left(0.438, 0.219, 0.146, 0.109, 0.088\right)$
(**2c***)	$\left(0.692, 0.231, 0.077\right)$	$\left(0.575, 0.230, 0.115, 0.057, 0.023\right)$
(**2d***)	$\left(0.776, 0.177, 0.047\right)$	$\left(0.684, 0.192, 0.078, 0.034, 0.012\right)$
(**3a**)	$\left(0.7, 0.2\overline{4}, 0.0\overline{5}\right)$	$\left(0.533, 0.266, 0.133, 0.055, 0.013\right)$
RE ($p = 2$): (**3c***)	$\left(0.643, 0.286, 0.071\right)$	$\left(0.\overline{45}, 0.2\overline{90}, 0.1\overline{63}, 0.0\overline{72}, 0.0\overline{18}\right)$
(**3d***)	$\left(0.7\overline{3}, 0.\overline{2}, 0.0\overline{4}\right)$	$\left(0.571, 0.257, 0.117, 0.045, 0.01\right)$
(**4a**)	$\left(0.7\overline{2}, 0.\overline{2}, 0.0\overline{5}\right)$	$\left(0.58, 0.24\overline{6}, 0.11\overline{3}, 0.04\overline{6}, 0.01\overline{3}\right)$
(**4c***)	$\left(0.6, 0.3, 0.1\right)$	$\left(0.428, 0.286, 0.171, 0.086, 0.029\right)$
(**4d***)	$\left(0.\overline{6}, 0.\overline{259}, 0.\overline{074}\right)$	$\left(0.5, 0.272, 0.143, 0.065, 0.02\right)$

As previously justified, if a non-zero ε value is desired in the constraints, the most suitable option is to use the value $\varepsilon^*/2$ or, equivalently, to using the weights resulting from averaging the weights obtained for $\varepsilon = 0$ and those obtained for $\varepsilon = \varepsilon^*$. Table 5 shows some weights obtained in this way, although only those in which the value of w_1 is not excessively large compared to the rest have been calculated. It is worth noting that the average of the weights obtained in cases (**4a**) and (**4b***) also allows achieving the ROC weights:[15]

$$\frac{1}{2}\left(\frac{2}{m}\sum_{k=j}^{m}\frac{k+1-j}{k(k+1)}+\frac{2(m+1-j)}{m(m+1)}\right)=\frac{1}{m}\left(\sum_{k=j}^{m}\frac{1}{k}-j\sum_{k=j}^{m}\left(\frac{1}{k}-\frac{1}{k+1}\right)+1-\frac{j}{m+1}\right)$$
$$=\frac{1}{m}\left(\sum_{k=j}^{m}\frac{1}{k}-j\left(\frac{1}{j}-\frac{1}{m+1}\right)+1-\frac{j}{m+1}\right)=\frac{1}{m}\sum_{k=j}^{m}\frac{1}{k}.$$

Table 5: Weighting vectors obtained as averages.

Cases	Weights ($m = 3$)	Weights ($m = 5$)
(**1a1b***)	$\left(0.47\overline{2}, 0.30\overline{5}, 0.\overline{2}\right)$	$\left(0.328, 0.228, 0.179, 0.145, 0.12\right)$
(**1a1c***)	$\left(0.\overline{5}, 0.30\overline{5}, 0.13\overline{8}\right)$	$\left(0.395, 0.262, 0.178, 0.112, 0.053\right)$
ROC: (**1a1d***), (**4a4b***)	$\left(0.6\overline{1}, 0.2\overline{7}, 0.\overline{1}\right)$	$\left(0.45\overline{6}, 0.25\overline{6}, 0.15\overline{6}, 0.09, 0.04\right)$
(**2a2b***)	$\left(0.6\overline{41}, 0.2\overline{37}, 0.1\overline{21}\right)$	$\left(0.532, 0.216, 0.122, 0.078, 0.052\right)$
(**3a3b***)	$\left(0.6, 0.2\overline{8}, 0.\overline{1}\right)$	$\left(0.433, 0.266, 0.167, 0.094, 0.04\right)$

It is also important to note that, although at first glance using weighting vectors that are the average of two others may seem unusual, the fact is that they allow obtaining weighting vectors that are very similar to those used in some sports competitions. For instance:

1. The scoring vector currently used in the Formula One World Championship is $(25, 18, 15, 12, 10, 8, 6, 4, 2, 1)$ ($m = 10$). If we calculate the vector in case (**3a3b***), divide it by w_m (so that the mth component of the vector equals 1), and round each weight to the nearest natural number,[16] the vector we obtain is $(26, 19, 15, 12, 9, 7, 5, 4, 2, 1)$.
2. The scoring vector used from 2003 to 2009 in the Formula One World Championship was $(10, 8, 6, 5, 4, 3, 2, 1)$ ($m = 8$). From the vector of case (**1a1c***) we get the vector $(13, 9, 7, 6, 4, 3, 2, 1)$.
3. The scoring vector used from 1991 to 2002 in the Formula One World Championship was $(10, 6, 4, 3, 2, 1)$ ($m = 6$). From the vector of case (**1a1c***) we get the vector $(9, 6, 5, 3, 2, 1)$.
4. The scoring vector used in The Best FIFA Football Awards is $(5, 3, 1)$ ($m = 3$). From the vector of case (**3a3b***) we get the vector $(5, 3, 1)$.

[15]We use the notation (**4a4b***) to indicate that the average has been taken between the weights obtained in cases (**4a**) and (**4b***). A similar comment can be applied to the rest of the cases.

[16]This procedure will be applied to all the following cases.

6. Concluding remarks

From the pioneering work of Cook & Kress (1990), various models have appeared in the field of preferential voting to deal with the problem of determining the scores associated with different ranking places. Although in the original model by Cook & Kress (1990) each candidate is evaluated with the scoring vector that favors her the most, the objective pursued in other models is to determine a single scoring vector for all candidates. In this paper we have analyzed two linear models proposed by Wang et al. (2007) (and the solutions of these models provided by Ahn (2017a)), as well as other alternative models suggested by Foroughi & Aouni (2012). The main drawback of all these models is that the uniqueness of the solutions cannot be guaranteed, so the winner or the final ranking of the candidates may depend on the chosen scoring vector. To avoid this problem, a possible solution is the use of surrogate weights, among which ROC weights stand out as the centroid of a specific simplex. Following this idea, in this paper we have provided the explicit expression for the weights that constitute the centroid of a wide variety of simplices used in ranking voting systems. Furthermore, we have justified that, in some cases, it makes sense to consider a scoring vector obtained as the average of two others, as the score obtained with it represents the average value of the scores obtained with the centroids of certain simplices that depend on a parameter ε. We have also seen that this procedure allows us to obtain weighting vectors that bear striking resemblance to those used in some sports competitions.

Finally, we would like to point out that, although some scoring rules are used in sports contests, they are rarely employed from a practical standpoint in the field of social choice, with the exception of the Borda rule and the plurality rule. However, they have been extensively studied and characterized in the literature (see, for instance, Smith (1973), Young (1975), Chebotarev & Shamis (1998), and Llamazares & Peña (2015b)). For this reason, it seems to us that it could be interesting, as a future line of research, to analyze, from the perspective of social choice, which properties satisfy the scoring rules generated from the weight vectors obtained in this work.

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