SING 2010
Resolution of Singularities Problems

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Titles and Abstracts

José Manuel Aroca (U. Valladolid)
Vincent Cossart and the singularities.

Angélica Benito (U. Autónoma Madrid)
Monoidal transforms and invariants of singularities in positive characteristic.

Joint work with O. Villamayor.

A typical problem in the area of resolution appears whenever we fix a hypersurface X, embedded in a smooth scheme, with points of multiplicity at most n. The problem is to define a sequence of monoidal transformations with centers included in the n-fold points of X (and its successive transforms) so that the final strict transform of X has no n-fold points. If X is defined over a field of characteristic 0, to each sequence of transformations, one can attach a monomial ideal supported on the exceptional locus. If some particular conditions hold (*monomial case*), the combinatorial resolution of this ideal implies resolution of singularities. In this talk, we present an analogous fact over fields of positive characteristic: we describe a monomial ideal whose resolution, under some particular conditions (*strong monomial case*), implies resolution of singularities. The definition of that monomial ideal requires the development of new invariants. These invariants give rise to new strategies (by stratification of the singular locus) to face the problem of resolution of singularities. In the case of schemes of dimension 2, the embedded resolution of singularities is achieved in a very synthetic manner, alternative to the more general recent result of Cossart–Jansen–Saito.

Steven Dale Cutkosky (U. Missouri)
Monomialization in higher dimensions.

Suppose that K is an algebraically closed field of characteristic zero. A morphism $\Phi : K^n \to K^m$ is *monomial* if there exists $a_{ij} \in \mathbb{N}$ such that $y_i = \prod_{j=1}^{n} x_i^{a_{ij}}$ for $1 \leqslant i \leqslant m$, where $x_1, \ldots, x_n$ are the coordinates of K^n and $y_1, \ldots, y_m$ are the coordinates of K^m.

Suppose that $\Phi : X \to Y$ is a dominant morphism of nonsingular K-varieties. Φ is *locally monomial* if for all $p \in X$, the germ of Φ at p is formally isomorphic to a germ of a monomial morphism (not necessarily at the origin). If global simple normal crossing divisors D_Y and $D_X = \Phi^{-1}(D_Y)$ exist which are compatible with these isomorphisms, then Φ is toroidal.

A *monomialization* (*toroidalization*) of Φ is a commutative diagram of morphisms

$$\begin{array}{ccc} X_1 & \xrightarrow{\Phi_1} & Y_1 \\ {\scriptstyle \Psi_1}\downarrow & & \downarrow{\scriptstyle \Psi_2} \\ X & \xrightarrow{\Phi} & Y \end{array}$$

such that Ψ_1, Ψ_2 are projective birational morphisms which are products of monoidal transforms (blow ups of nonsingular subvarieties) and Φ_1 is locally monomial (toroidal).

If Y is a curve, the construction of a monomialization (or toroidalization) is a consequence of resolution of singularities. However, new techniques are required when Y has higher dimension. There are several, relatively simple, proofs of monomialization and toroidalization when X and Y are surfaces. Unfortunately, these proofs do not readily extend to higher dimensions.

In previous work, the author has proven that a local monomialization can be constructed locally along a fixed but arbitrary valuation in arbitrary dimensions.

The author has also proven local monomialization and toroidalization when X is a 3-fold, and Y has arbitrary dimension ($\leqslant 3$).

In our local proof, invariants are used which measure the deviation of a morphism from being locally monomial, and behave well under blow ups. The invariants are tied to the rational rank of a particular valuation. The choice of centers is highly noncanonical, and is tied to the particular choice of valuation.

In our global proof in dimension 3, we found that natural invariants tend to behave very badly. For instance, the "kangaroo point" phenomenon can occur. This vexing problem also appears in resolution of singularities in characteristic p and in resolution of vector fields. At a kangaroo point, a natural invariant may go up by at most one after blowing up, but must eventually come down to at most the original value.

Recently, we have been able to globalize some of our local proof, above a fixed point of X such that $\Phi : X \to Y$ has been suitably prepared. Using this, we are able to prove local monomialization and toroidalization of dominant morphisms from a variety X of arbitrary dimension to a surface Y.

Samar ElHitti (NYC Coll. Tech.)
Formal prime ideals of infinite value and their algebraic resolution.

Suppose that R is a local domain essentially of finite type over a field of characteristic 0, and ν a valuation of the quotient field of R which dominates R. The rank of such a valuation often increases upon extending the valuation to a valuation dominating $\hat{R}$, the completion of R. When the rank of ν is 1, Cutkosky and Ghezzi handle this phenomenon by resolving the prime ideal of infinite value, but give an example showing that when the rank is greater than 1, there is no natural ideal in $\hat{R}$ that leads to this obstruction. We extend their result on the resolution of prime ideals of infinite value to valuations of arbitrary rank. (Joint work with S. Dale Cutkosky.)

Herwig Hauser (U. Wien)
Prime characteristic versus zero characteristic in the resolution of surfaces.

We show how the invariant for the resolution of surfaces in characteristic zero has to be understood so as to be applicable also in positive characteristic.

Heisuke Hironaka (JAMS)
(title to be announced)

Uwe Jannsen (U. Regensburg)
Strong resolution of singularities for two-dimensional schemes.

For varieties over fields of characteristic zero there are very strong results (by Hironaka and several others) on resolution of singularities: a canonical and to a certain extent also functorial procedure by so-called permissible blow-ups.

Over fields of positive characteristic or for schemes with mixed characteristic very little is known, and only in small dimension—more precisely in dimensions 2 and 3.

Even in dimension 2, where one has several methods (e.g., a procedure by Lipman using alternately normalization and blow-ups in points), there were until recently no results on strong resolution as described above.

I will report on joint work with V. Cossart and S. Saito, in which we fill this gap. We can even treat non-irreducible or non-reducible schemes, as well as "boundaries" on them.

Hiraku Kawanoue (RIMS)
On the Idealistic Filtration Program.

The Idealistic Filtration Program (IFP) is one of the approaches to resolution of singularities over an algebraically closed field of arbitrary characteristic, which I proposed and developed with K. Matsuki. In my talk, I will give an introduction to IFP and explain some partial results.

Kiran Kedlaya (MIT)
Resolution of turning points of formal flat meromorphic connections.

The local structure of formal flat meromorphic connections on a smooth complex algebraic or analytic variety was studied by Malgrange and Sabbah. One gets a good analogue of the Turrittin–Levelt–Hukuhara decomposition at each point away from a closed subvariety of codimension at least 2, called the turning locus. We establish that one can locally blow up in the turning locus so as to obtain good decompositions at all points. The obstruction to extending this construction to a global blowup seems to be a form of embedded resolution of singularities for certain virtual closed subschemes (nef b-divisors).

Franz-Viktor Kuhlmann (U. Saskatchewan)
Artin-Schreier Extensions and Elimination of Wild Ramification.

Elimination of wild ramification is used in the structure theory of valued function fields, with applications in areas such as local uniformization and the model theory of valued fields. I will give a survey on the role that Artin–Schreier extensions play in the elimination of wild ramification and corresponding main results on the structure of valued function fields. Further, I will describe a classification of Artin–Schreier extensions with non-trivial defect. It can be used to improve one of these main theorems ("Henselian Rationality"); this is recent work based on discussions with Michael Temkin. Finally, I will state several open problems.

Ben Lichtin

Monomialization of maps and multivariate asymptotics of fiber integrals.

We describe some applications of multivariate asymptotic analysis (MAA) to problems in harmonic and complex analysis. MAA is capable of describing with good precision the singular behavior of the fiber integral of an analytic mapping $F : \mathbb{R}^n \to \mathbb{R}^k$. It can therefore be understood as a natural extension of a classical technique applied to functions ($k = 1$).

The local monomialization of F is the key geometric ingredient for MAA. It is the natural analogue of local uniformization for a function, and can be used in a similar way to understand the fiber integral near the critical locus of F.

To illustrate the applications it is useful to have large classes of maps whose local monomializations are reasonably explicit. The case $k = n$ is of special interest since the critical locus is defined by a single function. As a result, Newton polyhedra can be used to describe the monomialization for generic maps.

We will then show how MAA helps to address problems of Stein and Berenstein–Yger, in particular. The first asks for an effective lower bound on p for which a certain maximal operator (associated to a nonsingular hypersurface) is bounded on $L^p(\mathbb{R}^n)$. The second asks for the largest possible class of paths over which an asymptotic approximation exists for a residue current, defined by a local complete intersection. If time permits, a third problem for which MAA is also useful will be discussed. This is the "stability of integrals" (in the sense of Phong–Stein–Sturm) within families of functions.

Daniel Panazzolo (U. Haute-Alsace)

Techniques pour la résolution des singularités d'un champ de vecteurs.

On présentera les outils utilisés dans la démonstration d'un théorème de résolution de singularités pour les champs de vecteurs en dimension trois.

Olivier Piltant (CNRS & U. Versailles)

Cossart's strategies for resolution in dimension three.

Cossart's resolution of singularities for purely inseparable coverings of a regular threefold (of characteristic $p > 0$) was extended in [CP] to prove resolution for algebraic threefolds defined over any function field $k(\lambda_1, \ldots, \lambda_s)$, with k_0 a perfect field. The proof brings together:

1) Ramification theoretic methods extending works of Abhyankar for surfaces to reduce resolution to local uniformization for Artin–Schreier or purely inseparable coverings of degree p of a regular threefold.

2) Global resolution methods for resolving such coverings, based on characteristic polyhedra and differential invariants.

I will explain the structure and main ingredients in the proof of 2), with a view to mixed characteristic (joint work with V. Cossart, in preparation).

[C] Cossart V., "Sur le polyèdre caractéristique", Thèse d'état, Orsay (1987).

[CP] Cossart, V. and Piltant, O., "Resolution of singularities of threefolds in positive characteristic. I, II.", *J. Algebra* **320** (2008), no. 3, 1051–1082 and *ibid.* **321** (2009), no. 7, 1836–1976.

Ana J. Reguera (U. Valladolid)

On divisorial valuations and the space of arcs.

Let X be an algebraic or formal variety over a perfect field, and let us assume the existence of a resolution of singularities of X. In the midsixties, J. Nash proposed to study the essential divisorial valuations of X (i.e., appearing modulo birational equivalence in every resolution of singularities of X) from the space of arcs of X. We will explain some of the main ideas that have been applied in the Nash problem. In particular, we will emphasize the underlying idea in our common work with V. Cossart [CPR].

[CPR] V. Cossart, O. Piltant, A.J. Reguera, *Divisorial valuations dominating rational surface singularities*, Fields Institute Communications series. Amer. Math. Soc. 32, 89–101 (2002).

Claude Roche (U. Toulouse)

Local Uniformisation of vector fields, three dimensional case.

In view of the birational reduction of singularities for one dimensional foliations in an ambient projective variety of dimension three, in a joint work with F. Cano and M. Spivakovsky, we establish the existence of a Local Uniformization in the sense of Zariski. The patching procedure, similar to the one given by Zariski in the case of varieties, of which an axiomatic adaptation was given by O. Piltant, gives our "Theorem of reduction of singularities in dimension three". After a *tour d'horizon*, we will discuss technicalities of the most peculiar situations.

Fernando Sanz (U. Valladolid)
Monomialization of convergent generalized power series and subanalytic geometry.
In this talk, we present the project of Ph.D. Thesis of Rafael Martin, a speaker's student (together with J.-P. Rolin as co-advisor). It deals with the study of the subanalytic geometry associated with the ring of real convergent generalized power series (convergent power series for which the exponents in each variable is a special well ordered subset of the real field). The aim is to prove a Hironaka's Rectilinearization Theorem in this framework. We discuss here one step contributing to this program: the monomialization of generalized power series by means of blowing-ups.

Mark Spivakovsky (CNRS & U. Toulouse)
On local uniformization of an equicharacteristic quasi-excellent local domain whose residue field k ***satisfies*** $[k : k^p] < \infty$***.***
The purpose of this lecture is to report on our recent results regarding local uniformization in positive characteristic. We will use two types of valuation-theoretic tools: key polynomials (particularly, limit key polynomials in positive characteristic and their behaviour under differential operators) and generalized Puiseux expansion.

Bernard Teissier (CNRS & Inst. Math. Jussieu)
A viewpoint on local resolution.

Michael Temkin (IAS)
Inseparable local uniformization.
It is known since the works of Zariski in early 40ies that desingularization of schemes along valuations (called local uniformization of valuations) can be considered as the local part of the desingularization problem. It is still an open problem if local uniformization exists for varieties of positive characteristic and dimension larger than three. The main result I will discuss in this lecture is that local uniformization of varieties exists after a purely inseparable extension of the field of rational functions, i.e., any valuation can be uniformized by a purely inseparable alteration. If time permits I will also discuss conjectural generalizations to all quasi-excellent schemes (including the mixed characteristic case) which, as I expect, can be proven by a similar method.

Jaroslaw Wlodarczyk (U. Purdue)
(title to be announced)